BACKPAPER: RINGS AND MODULES

Date: **June 2026**
The Total points is **100**.

A **ring** would mean a **commutative ring with identity**.

(1) (7+7+7=21 points) Prove or disprove the following.
   (a) If R_1 and R_2 are PID then so is $R_1 \times R_2$ is PID.
   (b) If R_1 is noetherian and R_2 is a subring of R_1 then R_2 is noetherian.
   (c) If $q : R_1 \to R_2$ is a surjective ring homomorphism and R_1 is PID then R_2 is noetherian.

(2) (8+8=16 points) Let $R = (\mathbb{Z}/180\mathbb{Z})[x]$ be the polynomial ring over $\mathbb{Z}/180\mathbb{Z}$. Compute the nilradical of R. Prove that every nonzero $(\mathbb{Z}/180\mathbb{Z})[x]$-module contains a nonzero torsion element.

(3) (7+7+7=21 points) Give a counter example to disprove the following statements.
   (a) Every Unique factorization domain is a Principal ideal domain.
   (b) For a ring R, every torsion module has a nonzero annihilator.
   (c) Every commutative ring with unity is noetherian.

(4) (10+10=20 points) Consider the following rings.
   (a) $A = (\mathbb{Z}/45\mathbb{Z})[x]/(3x - 1)$
   (b) $B = (\mathbb{Z}/5\mathbb{Z})[x]/(3x - 1)$
   (c) $C = \mathbb{Z}[x]/(3x - 1)$

   Show that A and B are isomorphic. Show that these rings are PID and determine if any of them are a field.

(5) (10+12=22 points) Let $V_1 = \mathbb{C}[y]/(y(y^2 - 1)(y^3 - 1))$, $V_2 = \mathbb{C}[z]/(z^3 - z^2)$ and $V = V_1 \oplus V_2$ be $\mathbb{C}$-vector spaces. Let $\phi_1 : V_1 \to V_1$, $\phi_2 : V_2 \to V_2$ and $\phi : V \to V$ be given by $\phi_1(v_1) = \bar{y}v_1$, $\phi_2(v_2) = \bar{z}v_2$ and $\phi(v_1, v_2) = (\bar{y}v_1, \bar{z}v_2)$ $\forall v_1 \in V_1, \forall v_2 \in V_2$ respectively. Compute the minimal polynomial and the characteristic polynomial of ϕ_1, ϕ_2 and ϕ. Find the rational canonical form and the Jordan form of ϕ?